

# Hom-Lie-Yamaguti Structures on Hom-Leibniz Algebras

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## Abstract

Every multiplicative left Hom-Leibniz algebra has a natural Hom-Lie-Yamaguti structure.

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## 1. Introduction and statement of result

A (left) *Leibniz algebra* is an algebra  $(L, \cdot)$  satisfying the identity

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z + y \cdot (x \cdot z).$$

Leibniz algebras were introduced by J.-L. Loday [15] (and so they are sometimes called Loday algebras) as a noncommutative analogue of Lie algebras, in the study of some topics in homological algebra and noncommutative geometry (see also [5], [20]). While earlier papers on Leibniz algebras are concerned with some homological problems (see, e.g., [15], [20]), some structure theory of Leibniz algebras are proposed in, e.g., [2] and [3] (see also references therein). Classification of low-dimensional Leibniz algebras could be found in, e.g., [2], [6], [15], [21].

One of the problems in the general theory of a given class of (binary or binary-ternary) nonassociative algebras is the study of relationships between that class of algebras and the one of Lie algebras. In the same rule, the search of relationships between a class of nonassociative

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algebras and the one of Leibniz algebras is of interest (at least for constructing concrete examples of the given class of nonassociative algebras). In this setting, the existence of a Lie-Yamaguti structure on any (left) Leibniz algebra pointed out in [14] is a good illustration.

A *Lie-Yamaguti algebra* is a triple  $(L, [, ], \{, \}, \cdot)$  in which  $L$  is a vector space,  $[, ] : L \times L \rightarrow L$  a bilinear map and  $\{, \}, \cdot : L \times L \times L \rightarrow L$  a trilinear map such that

$$\begin{aligned} \text{(LY1)} \quad & [x, y] = -[y, x], \\ \text{(LY2)} \quad & \{x, y, z\} = -\{y, x, z\}, \\ \text{(LY3)} \quad & \circ_{x,y,z} ([x, y], z) + \{x, y, z\} = 0, \\ \text{(LY4)} \quad & \circ_{x,y,z} \{[x, y], z, u\} = 0, \\ \text{(LY5)} \quad & \{x, y, [u, v]\} = [\{x, y, u\}, v] + [u, \{x, y, v\}], \\ \text{(LY6)} \quad & \{x, y, \{u, v, w\}\} = \{\{x, y, u\}, v, w\} + \{u, \{x, y, v\}, w\} \\ & \quad + \{u, v, \{x, y, w\}\}, \end{aligned}$$

for all  $u, v, w, x, y, z$ , in  $L$ , where  $\circ_{x,y,z}$  denotes the sum over cyclic permutation of  $x, y, z$ . Lie-Yamaguti algebras, first called "generalized Lie triple systems", were introduced by K. Yamaguti [22] while giving an algebraic interpretation of the characteristic properties of the torsion and curvature of a homogeneous space with canonical connection (the Nomizu's connection) [19]. Later, M. Kikkawa [13] called them "Lie triple algebras" and the terminology of "Lie-Yamaguti algebras" is introduced in [14] to designate these algebras. As in [4], we write "LY-algebras" for Lie-Yamaguti algebras.

In a left Leibniz algebra  $(L, \cdot)$  if define  $[x, y] := x \cdot y - y \cdot x$  (skew-symmetrization) and  $\{x, y, z\} := -(x \cdot y) \cdot z$ , then  $(L, [, ], \{, \}, \cdot)$  is a LY-algebra [14]. In this note, we will be interested in the counterpart of this construction in the Hom-algebra setting.

Roughly speaking, a Hom-type generalization of a kind of algebras is defined by twisting its defining identities by a linear self-map (the twisting map) in such a way that when the twisting map is the identity map, one recovers the original kind of algebras. The theory of Hom-algebras originated from the introduction of so-called "Hom-Lie algebras" in [8] as an abstraction of an approach to deformations of the Witt algebra and the Virasoro algebra based on  $\sigma$ -derivations, including  $q$ -derivations of the Witt algebra and the Virasoro algebra associated to  $q$ -difference operators. The outcome of the algebraic view of these considerations is the introduction of "Hom-associative algebras" in [18] as a Hom-analogue of associative algebras. Hom-associative algebras are for Hom-Lie algebras what are associative algebras for Lie algebras: the commutator-algebra of a Hom-associative algebra is a Hom-Lie algebra [18]. As a "noncommutative" generalization of Hom-Lie algebras, Hom-Leibniz algebras are also defined in [18], where other results regarding Hom-algebras are found (see also [23]). A general method of twisting ordinary algebras into their Hom-type analogues is given in [24]. The reader is referred to, e.g., [7], [11], [16], [17], [25] for discussions about various Hom-type algebras.

Following this line, Hom-Lie-Yamaguti algebras (Hom-LY algebras) are introduced in [7]. A *Hom-Lie-Yamaguti algebra* (Hom-LY algebra for short) is a quadruple  $(L, [, ], \{, \}, \alpha)$  in which  $L$  is a vector space,  $[, ]$  a binary operation and  $\{, \}, \cdot$  a ternary operation on  $L$ , and  $\alpha : L \rightarrow L$  a linear map such that

$$\begin{aligned} \text{(HLY1)} \quad & \alpha([x, y]) = [\alpha(x), \alpha(y)], \\ \text{(HLY2)} \quad & \alpha(\{x, y, z\}) = \{\alpha(x), \alpha(y), \alpha(z)\}, \\ \text{(HLY3)} \quad & [x, y] = -[y, x], \\ \text{(HLY4)} \quad & \{x, y, z\} = -\{y, x, z\}, \\ \text{(HLY5)} \quad & \circ_{x,y,z} ([x, y], \alpha(z)) + \{x, y, z\} = 0, \end{aligned}$$

$$\begin{aligned}
(\text{HLY6}) \quad & \circlearrowleft_{x,y,z} \{[x,y], \alpha(z), \alpha(u)\} = 0, \\
(\text{HLY7}) \quad & \{\alpha(x), \alpha(y), [u,v]\} = [\{x,y,u\}, \alpha^2(v)] + [\alpha^2(u), \{x,y,v\}], \\
(\text{HLY8}) \quad & \{\alpha^2(x), \alpha^2(y), \{u,v,w\}\} = \{\{x,y,u\}, \alpha^2(v), \alpha^2(w)\} \\
& \quad + \{\alpha^2(u), \{x,y,v\}, \alpha^2(w)\} + \{\alpha^2(u), \alpha^2(v), \{x,y,w\}\},
\end{aligned}$$

for all  $u, v, w, x, y, z \in Y$ . Thus, as mentioned above, we shall prove the following

**Theorem.** *Every multiplicative left Hom-Leibniz algebra has a natural Hom-Lie-Yamaguti structure.*

The useful definitions and some facts as the characterization of the Hom-Akivis algebra associated to a given Hom-Leibniz algebra are reminded in section 2. In section 3, we prove the theorem and discuss examples of Hom-LY algebras that are constructed using the theorem above (thusly, we also construct examples of left Hom-Leibniz algebras).

All vector spaces and algebras are considered over a fixed ground field of characteristic 0.

## 2. Definitions and basic facts

We recall some basic notions, introduced in [8], [11], [18], [23], [24], related to Hom-algebras. We also recall from [12] a characterization of the Hom-Akivis algebra associated with a given Hom-Leibniz algebra.

**Definition 2.1.** ([18], [23]) A *Hom-algebra* is a triple  $(L, \cdot, \alpha)$  in which  $L$  is a vector space, " $\cdot$ " a binary operation on  $L$  and  $\alpha : L \rightarrow L$  is a linear map (the twisting map).

A Hom-algebra  $(L, \cdot, \alpha)$  is said to be *multiplicative* if  $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$  (multiplicativity), for all  $x, y$  in  $L$ .

Since our result depends on multiplicativity, we assume here that all Hom-algebras are multiplicative.

**Definition 2.2.** Let  $(L, \cdot, \alpha)$  be a Hom-algebra.

(i) The *Hom-associator* [18] of  $L$  is the trilinear map  $as_\alpha : L \times L \times L \rightarrow L$  defined by

$$as_\alpha(x, y, z) = (x \cdot y) \cdot \alpha(z) - \alpha(x) \cdot (y \cdot z), \quad (2.1)$$

for all  $x, y, z \in L$ . If  $as_\alpha(x, y, z) = 0$  (Hom-associativity),  $\forall x, y, z \in L$ , then  $(L, \cdot, \alpha)$  is said to be *Hom-associative* [18].

(ii) The *Hom-Jacobian* [18] of  $L$  is the trilinear map  $J_\alpha : L \times L \times L \rightarrow L$  defined by

$$J_\alpha(x, y, z) := \circlearrowleft_{x,y,z} (x \cdot y) \cdot \alpha(z) \quad (2.2)$$

for all  $x, y, z$  in  $L$ . The Hom-algebra  $(L, \cdot, \alpha)$  is called a *Hom-Lie algebra* [8] if the operation " $\cdot$ " is anticommutative and the *Hom-Jacobi identity*  $J_\alpha(x, y, z) = 0$  is satisfied in  $(L, \cdot, \alpha)$ .

**Remark 2.3.** If  $\alpha = Id$  (the identity map) then (2.1) (resp. (2.2)) is just the associator (resp. the Jacobian) in  $(L, \cdot, \alpha)$ . Therefore an associative (resp. a Lie) algebra could be seen as a Hom-associative (resp. Hom-Lie) algebra with the identity map as the twisting map. Also note that a not necessarily Hom-associative algebra is called a *non-Hom-associative* algebra

in [11] in analogy with the case of not necessarily associative algebras (the terminologies of "Hom-nonassociative algebras" or "nonassociative Hom-algebras" are also used in [17], [23] for that type of Hom-algebras).

As for Lie algebras, Hom-Lie algebras have a "noncommutative" generalization as Hom-Leibniz algebras.

**Definition 2.4.** ([18]) A *(left) Hom-Leibniz algebra* is a Hom-algebra  $(L, \cdot, \alpha)$  satisfying the *(left) Hom-Leibniz identity*

$$\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \alpha(z) + \alpha(y) \cdot (x \cdot z) \quad (2.3)$$

for all  $x, y, z$  in  $L$ .

**Remark 2.5.** If  $\alpha = Id$  in Definition 2.4, then  $(L, \cdot, \alpha)$  reduces to a (left) Leibniz algebra  $(L, \cdot)$ . Moreover, as for Leibniz algebras [15], if the operation of a given Hom-Leibniz algebra  $(L, \cdot, \alpha)$  is skew-symmetric (i.e. anticommutative), then  $(L, \cdot, \alpha)$  turns out to be a Hom-Lie algebra (see [18]). We also observe that the original definition of a Hom-Leibniz algebra [18] is related to the identity (the "right" Hom-Leibniz identity)

$$(x \cdot y) \cdot \alpha(z) = (x \cdot z) \cdot \alpha(y) + \alpha(x) \cdot (y \cdot z).$$

Moreover, given a linear self-map  $\alpha$  of  $L$ , every Leibniz algebra  $(L, \cdot)$  can be twisted into a Hom-Leibniz algebra  $(L, \dot{\alpha}, \alpha)$  with " $\dot{\alpha}$ " defined by  $x \dot{\alpha} y = \alpha(x \cdot y)$  for all  $x, y$  in  $L$  ([24]).

The extension to binary-ternary algebras of twisting identities of algebras is considered in [11]. This led to the introduction of the class of "Hom-Akivis algebras" as a twisted version of Akivis algebras introduced by M.A. Akivis (see [1] and references therein).

Akivis algebras (first called "W-algebras" [1]) arose in the differential geometry of differentiable webs, and also as tangent algebras to local differentiable quasigroups. The terminology of "Akivis algebras" is introduced in [9].

**Definition 2.6.** ([11]) A *Hom-Akivis algebra* is a quadruple  $(L, [, ], [, ], \alpha)$  such that  $L$  is a vector space, " $[,]$ " is a skew-symmetric binary operation on  $L$ , " $[[, ], ]$ " a ternary operation on  $L$ ,  $\alpha : L \rightarrow L$  a linear map, and that the *Hom-Akivis identity*

$$J_\alpha(x, y, z) = \odot_{x,y,z} [x, y, z] - \odot_{x,y,z} [y, x, z] \quad (2.4)$$

holds for all  $x, y, z$  in  $L$ .

Observe that for  $\alpha = Id$  the Hom-Akivis identity (2.4) reduces to the *Akivis identity*

$$J(x, y, z) = \odot_{x,y,z} [x, y, z] - \odot_{x,y,z} [y, x, z]$$

which defines *Akivis algebras*. It is shown [11] that every Akivis algebra with a linear self-map is twisted into a Hom-Akivis algebra and that every non-Hom-associative algebra with a linear self-map is a Hom-Akivis algebra with respect to the skew-symmetrization  $[x, y] = xy - yx$

and Hom-associator  $[x, y, z] = as_\alpha(x, y, z)$ .

In terms of Hom-associators, the identity (2.3) has the form

$$as_\alpha(x, y, z) = -\alpha(y) \cdot (x \cdot z). \quad (2.5)$$

Thus the operations of the Hom-Akivis algebra associated with the Hom-Leibniz algebra  $(L, \cdot, \alpha)$  are the skew-symmetrization and (2.5). Then the Hom-Akivis identity (2.4) takes the form

$$\circlearrowleft_{x,y,z} [[x, y], \alpha(z)] = \circlearrowleft_{x,y,z} as_\alpha(x, y, z) - \circlearrowleft_{x,y,z} as_\alpha(y, x, z)$$

that is, by (2.5) and (2.3),

$$\circlearrowleft_{x,y,z} [[x, y], \alpha(z)] = \circlearrowleft_{x,y,z} (x \cdot y) \cdot \alpha(z). \quad (2.6)$$

The considerations above will be used in the next section in the proof of the theorem.

### 3. Proof of the Theorem. Examples

In this section, we settle down in the proof of our claim, i.e. the existence of a Hom-LY structure on any (multiplicative) left Hom-Leibniz algebra. This proof is based on a specific ternary operation that can be considered on a given Hom-Leibniz algebra (this product is the Hom-analogue of the ternary operation considered in [14] on a left Leibniz algebra  $L$  that produces, along with the skew-symmetrization, a LY structure on  $L$ ). Also note that our proof below essentially relies on some properties characterizing Hom-Leibniz algebras, obtained in [12]. We conclude, as an illustration of our result, by some constructions of Hom-LY algebras from twisted Leibniz algebras (incidentally, this produces examples of left Hom-Leibniz algebras).

In a left Hom-Leibniz algebra  $(L, \cdot, \alpha)$  consider the skew-symmetrization

$$[x, y] := x \cdot y - y \cdot x$$

for all  $x, y$  in  $L$ . Then, from [12], we know that

$$(x \cdot y + y \cdot x) \cdot \alpha(z) = 0, \quad (3.1)$$

$$\alpha(x) \cdot [y, z] = [(x \cdot y), \alpha(z)] + [\alpha(y), (x \cdot z)]. \quad (3.2)$$

If consider the left translations  $\Lambda_a b := a \cdot b$  in  $(L, \cdot, \alpha)$ , then the identities (2.3) and (3.2) can be written respectively as

$$\Lambda_{\alpha(x)}(y \cdot z) = (\Lambda_x y) \cdot \alpha(z) + \alpha(y) \cdot (\Lambda_x z), \quad (3.3)$$

$$\Lambda_{\alpha(x)}[y, z] = [\Lambda_x y, \alpha(z)] + [\alpha(y), \Lambda_x z]. \quad (3.4)$$

*Proof of the Theorem.*

In  $(L, \cdot, \alpha)$  consider the following ternary operation:

$$\{x, y, z\} := as_\alpha(y, x, z) - as_\alpha(x, y, z) \quad (3.5)$$

for all  $x, y, z$  in  $L$ . Then (3.5), (2.3) and (2.5) imply

$$\{x, y, z\} = -(x \cdot y) \cdot \alpha(z). \quad (3.6)$$

Moreover, we have

$$\begin{aligned} [x, y] \cdot \alpha(z) &= (x \cdot y - y \cdot x) \cdot \alpha(z) \\ &= 2(x \cdot y) \cdot \alpha(z) \quad (\text{by (3.1)}) \\ &= -2\{x, y, z\} \quad (\text{see (3.6)}) \end{aligned}$$

so that

$$\{x, y, z\} = -\frac{1}{2}[x, y] \cdot \alpha(z). \quad (3.7)$$

Thus (3.5), (3.6) and (3.7) are different expressions of the operation " $\{, , \}$ " that are for use in what follows.

Now we proceed to verify the validity on  $(L, \cdot, \alpha)$  of the set of identities (HLY1)-(HLY8).

The multiplicativity of  $(L, \cdot, \alpha)$  implies (HLY1) and (HLY2) while (HLY3) is the skew-symmetrization and (HLY4) clearly follows from (3.5) (or (3.7)). Next, observe that (HLY5) is just the Hom-Akivis identity (2.6) for  $(L, \cdot, \alpha)$ .

Consider now  $\circlearrowleft_{(x,y,z)} \{[x, y], \alpha(z), \alpha(u)\}$ . Then

$$\begin{aligned} \circlearrowleft_{x,y,z} \{[x, y], \alpha(z), \alpha(u)\} &= \circlearrowleft_{x,y,z} -([x, y] \cdot \alpha(z)) \cdot \alpha^2(u) \quad (\text{by (3.6)}) \\ &= 2(\circlearrowleft_{x,y,z} \{x, y, z\}) \cdot \alpha^2(u) \quad (\text{by (3.7)}) \\ &= -2((x \cdot y) \cdot \alpha(z) + (y \cdot z) \cdot \alpha(x) + (z \cdot x) \cdot \alpha(y)) \cdot \alpha^2(u) \\ &= -2(\alpha(x) \cdot (y \cdot z) - \alpha(y) \cdot (x \cdot z) + (y \cdot z) \cdot \alpha(x) \\ &\quad + (z \cdot x) \cdot \alpha(y)) \cdot \alpha^2(u) \quad (\text{by (2.3)}) \\ &= -2(\alpha(x) \cdot (y \cdot z) + (y \cdot z) \cdot \alpha(x)) \cdot \alpha^2(u) \\ &\quad - 2(-\alpha(y) \cdot (x \cdot z) + (z \cdot x) \cdot \alpha(y)) \cdot \alpha^2(u) \\ &= -2(-\alpha(y) \cdot (x \cdot z) + (z \cdot x) \cdot \alpha(y)) \cdot \alpha^2(u) \quad (\text{by (3.1)}) \\ &= -2(-\alpha(y) \cdot (x \cdot z) - (x \cdot z) \cdot \alpha(y)) \cdot \alpha^2(u) \quad (\text{by (3.1)}) \\ &= 2(\alpha(y) \cdot (x \cdot z) + (x \cdot z) \cdot \alpha(y)) \cdot \alpha^2(u) \\ &= 0 \quad (\text{by (3.1)}) \end{aligned}$$

so that we get (HLY6). Next

$$\begin{aligned} \{\alpha(x), \alpha(y), [u, v]\} &= -\alpha(x \cdot y) \cdot \alpha([u, v]) \quad (\text{by (3.6) and multiplicativity}) \\ &= \Lambda_{-\alpha(x \cdot y)}[\alpha(u), \alpha(v)] \\ &= [\Lambda_{-x \cdot y} \alpha(u), \alpha^2(v)] + [\alpha^2(u), \Lambda_{-x \cdot y} \alpha(v)] \quad (\text{by (3.4)}) \\ &= [\{x, y, u\}, \alpha^2(v)] + [\alpha^2(u), \{x, y, v\}] \quad (\text{by (3.6)}) \end{aligned}$$

which is (HLY7). Finally, we compute

$$\begin{aligned}
& \{\{x, y, u\}, \alpha^2(v), \alpha^2(w)\} + \{\alpha^2(u), \{x, y, v\}, \alpha^2(w)\} \\
& + \{\alpha^2(u), \alpha^2(v), \{x, y, w\}\} \\
= & \{-\Lambda_{x \cdot y} \alpha(u), \alpha^2(v), \alpha^2(w)\} + \{\alpha^2(u), -\Lambda_{x \cdot y} \alpha(v), \alpha^2(w)\} \\
& + \{\alpha^2(u), \alpha^2(v), -\Lambda_{x \cdot y} \alpha(w)\} \\
= & -((-\Lambda_{x \cdot y} \alpha(u)) \cdot \alpha^2(v)) \cdot \alpha^3(w) - (\alpha^2(u) \cdot (-\Lambda_{x \cdot y} \alpha(v))) \cdot \alpha^3(w) \\
& - (\alpha^2(u) \cdot \alpha^2(v)) \alpha(-\Lambda_{x \cdot y} \alpha(w)) \\
= & (\Lambda_{\alpha(x \cdot y)} \alpha(u \cdot v)) \cdot \alpha^3(w) + \alpha^2(u \cdot v) \cdot \Lambda_{\alpha(x \cdot y)} \alpha^2(w) \quad (\text{by (3.3) and multiplicativity}) \\
= & \Lambda_{\alpha^2(x \cdot y)} (\alpha(u \cdot v) \cdot \alpha^2(w)) \quad (\text{by (3.3)}) \\
= & -\alpha^2(x \cdot y) \cdot (-\alpha(u \cdot v) \cdot \alpha^2(w)) \\
= & -(\alpha^2(x) \cdot \alpha^2(y)) \cdot \alpha(-(u \cdot v) \cdot \alpha(w)) \quad (\text{by multiplicativity}) \\
= & \{\alpha^2(x), \alpha^2(y), \{u, v, w\}\} \quad (\text{by (3.6)}).
\end{aligned}$$

Therefore  $(L, [, ], \{, \}, \alpha)$  is a Hom-LY algebra. This completes the proof.  $\square$

**Remark 3.1.** If set  $\alpha = Id$  in (3.6), then we recover the ternary operation defined in [14] in the proof of the existence of a natural LY structure on any left Leibniz algebra (see section 1). Therefore, although with a quite different scheme of proof, our result here is an  $\alpha$ -twisted version of the one in [14]. The untwisted version of the proof proposed here could be found in [10], where the result of [14] is considered again but via Akiwis algebras.

We now discuss examples of Hom-LY algebras that can be constructed using the theorem. Examples of Hom-LY algebras constructed from LY algebras can be found in [7].

**Example 3.2.** Let  $(L, \cdot, \alpha)$  be an anticommutative multiplicative left Hom-Leibniz algebra. Then  $(L, \cdot, \alpha)$  is a multiplicative Hom-Lie algebra ([18]). If define on  $L$  a ternary operation “ $\{, \cdot, \cdot\}$ ” by (3.6), then one checks that  $(L, \cdot, \{, \cdot, \cdot\}, \alpha)$  is a Hom-LY algebra.

In the following examples, the unspecified products are regarded as zero.

**Example 3.3.** Let  $(L, \cdot)$  be a 3-dimensional complex algebra defined by

$$e_2 \cdot e_3 = e_2, e_3 \cdot e_1 = \lambda e_1, e_3 \cdot e_2 = -e_2, e_3 \cdot e_3 = e_1 \quad (\lambda \in \mathbb{C}).$$

Then  $(L, \cdot)$  is a (solvable) complex left Leibniz algebra ([21], Table 3, algebra  $L_4$ ). Now a few computation shows that all the linear self-maps  $\alpha$  of  $L$  given by

$$\alpha(e_1) = (a\lambda + 1)e_1, \alpha(e_2) = be_2, \alpha(e_3) = ae_1 + e_2 + e_3$$

are endomorphisms of  $(L, \cdot)$ , where  $a, b, \lambda \in \mathbb{C}$ . By a result in [24], we define on  $L$  an operation “ $\dot{\alpha}$ ” by

$$e_2 \dot{\alpha} e_3 := \alpha(e_2 \cdot e_3) = be_2,$$

$$\begin{aligned}
e_3 \dot{\alpha} e_1 &:= \alpha(e_3 \cdot e_1) = \lambda(a\lambda + 1)e_1, \\
e_3 \dot{\alpha} e_2 &:= \alpha(e_3 \cdot e_2) = -be_2, \\
e_3 \dot{\alpha} e_3 &:= \alpha(e_3 \cdot e_3) = (a\lambda + 1)e_1
\end{aligned}$$

to get a multiplicative left Hom-Leibniz algebra  $L_\alpha := (L, \dot{\alpha}, \alpha)$ . Therefore, according to the theorem, the Hom-LY algebra  $(L, [, ], \{, \}, \alpha)$  corresponding to  $L_\alpha$  is defined by

$$\begin{aligned}
[e_1, e_3] &= -\lambda(a\lambda + 1)e_1 \quad (= -[e_3, e_1]), \\
[e_2, e_3] &= 2be_2 \quad (= -[e_3, e_2]), \\
\{e_3, e_2, e_3\} &= b^2e_2 \quad (= -\{e_2, e_3, e_3\})
\end{aligned}$$

with  $a, b, \lambda \in \mathbb{C}$ .

**Example 3.4.** Let  $(L, \cdot)$  be a 4-dimensional complex algebra defined by

$$\begin{aligned}
e_1 \cdot e_1 &= e_4, \quad e_1 \cdot e_2 = e_3, \quad e_1 \cdot e_3 = e_4, \\
e_2 \cdot e_1 &= -e_3, \\
e_3 \cdot e_1 &= -e_4.
\end{aligned}$$

Then  $(L, \cdot)$  is a (nilpotent) complex left Leibniz algebra ([2], Theorem 3.2, algebra  $\mathfrak{R}_7$ ). If define a linear self-map of  $L$  by

$$\begin{aligned}
\alpha(e_1) &= e_1 + e_2 + e_3 + e_4, \\
\alpha(e_2) &= e_2 + e_3 + e_4, \\
\alpha(e_3) &= e_3 + e_4, \\
\alpha(e_4) &= e_4,
\end{aligned}$$

then  $\alpha$  is verified to be an endomorphism of  $(L, \cdot)$ . Next, as in the example above, we may define on  $L$  an operation “ $\dot{\alpha}$ ” by

$$\begin{aligned}
e_1 \dot{\alpha} e_1 &= e_4, \quad e_1 \dot{\alpha} e_2 = e_3 + e_4, \quad e_1 \dot{\alpha} e_3 = e_4, \\
e_2 \dot{\alpha} e_1 &= -e_3 - e_4, \\
e_3 \dot{\alpha} e_1 &= -e_4
\end{aligned}$$

and then  $L_\alpha := (L, \dot{\alpha}, \alpha)$  is a multiplicative left Hom-Leibniz algebra. Now, applying the theorem, we see that the Hom-LY algebra  $(L, [, ], \{, \}, \alpha)$  corresponding to  $L_\alpha$  is constructed by defining

$$\begin{aligned}
[e_1, e_2] &= 2(e_3 + e_4) \quad (= -[e_2, e_1]), \quad [e_1, e_3] = 2e_4 \quad (= -[e_3, e_1]), \\
\{e_1, e_2, e_1\} &= e_4 \quad (= -\{e_2, e_1, e_1\}).
\end{aligned}$$

**Example 3.5.** A 4-dimensional (nilpotent) complex left Leibniz algebra  $(L, \cdot)$  is given by

$$\begin{aligned}
e_1 \cdot e_1 &= e_4, \quad e_1 \cdot e_2 = e_3, \quad e_1 \cdot e_3 = e_4, \\
e_2 \cdot e_1 &= -e_3 + e_4,
\end{aligned}$$



$$e_3 \cdot e_1 = -e_4$$

(see [2], Theorem 3.2, algebra  $\mathfrak{R}_8$ ). A linear self-map  $\alpha$  of  $L$  defined by

$$\begin{aligned}\alpha(e_1) &= e_1 + e_3 + e_4, \\ \alpha(e_2) &= e_2 + e_4, \\ \alpha(e_3) &= e_3, \\ \alpha(e_4) &= e_4,\end{aligned}$$

is verified to be an endomorphism of  $(L, \cdot)$ . Again, as in the examples above, define on  $L$  an operation “ $\dot{\alpha}$ ” by

$$\begin{aligned}e_1 \dot{\alpha} e_1 &= e_4, \quad e_1 \dot{\alpha} e_2 = e_3, \quad e_1 \dot{\alpha} e_3 = e_4, \\ e_2 \dot{\alpha} e_1 &= -e_3 + e_4, \\ e_3 \dot{\alpha} e_1 &= -e_4\end{aligned}$$

and then  $L_\alpha := (L, \dot{\alpha}, \alpha)$  is a multiplicative left Hom-Leibniz algebra. So the theorem implies that the Hom-LY algebra  $(L, [, ], \{, \}, \alpha)$  constructed from  $L_\alpha$  is defined by

$$\begin{aligned}[e_1, e_2] &= 2e_3 - e_4 \quad (= -[e_2, e_1]), \quad [e_1, e_3] = 2e_4 \quad (= -[e_3, e_1]), \\ \{e_1, e_2, e_1\} &= e_4 \quad (= -\{e_2, e_1, e_1\}).\end{aligned}$$

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